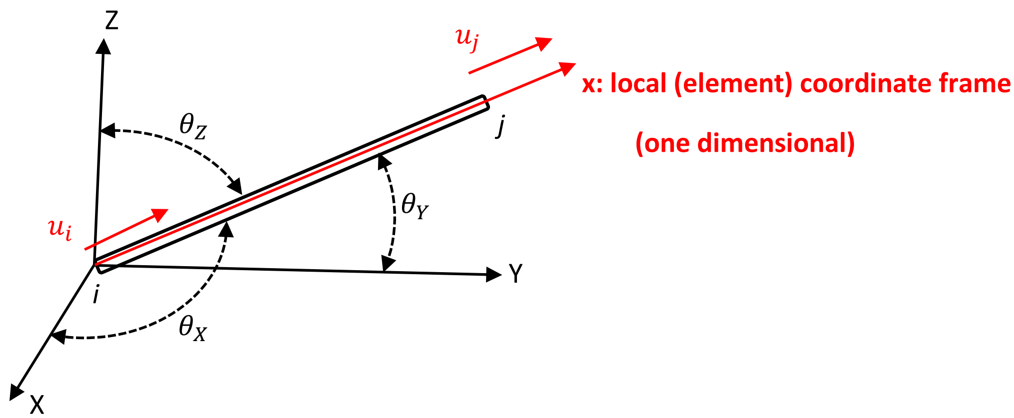
ME 402 – INTRODUCTION TO FINITE ELEMENT ANALYSIS (2014-2015 Spring)

HOMEWORK #1 - Solutions

Solution 1. (Chapter 2, Problem 18)



Displacements of nodes i and j on Global Coordinate Frame can be written in terms of local displacements as;

$$U_{iX} = u_{ix} * \cos\theta_X$$

$$U_{iY} = u_{ix} * \cos\theta_Y$$

$$U_{iZ} = u_{ix} * \cos\theta_Z$$

$$U_{jX} = u_{jx} * \cos\theta_X$$

$$U_{jY} = u_{jx} * \cos\theta_Y$$

$$U_{jZ} = u_{jx} * \cos\theta_Z$$

Matrix form of these equations;

$$\begin{Bmatrix} U_{iX} \\ U_{iY} \\ U_{iZ} \\ U_{jX} \\ U_{jY} \\ U_{jZ} \end{Bmatrix} = \begin{bmatrix} \cos\theta_X & 0 \\ \cos\theta_Y & 0 \\ \cos\theta_Z & 0 \\ 0 & \cos\theta_X \\ 0 & \cos\theta_Y \\ 0 & \cos\theta_Z \end{bmatrix} * \begin{Bmatrix} u_{ix} \\ u_{jx} \end{Bmatrix} \quad \Rightarrow \quad \boldsymbol{U} = \boldsymbol{T} * \boldsymbol{u}$$

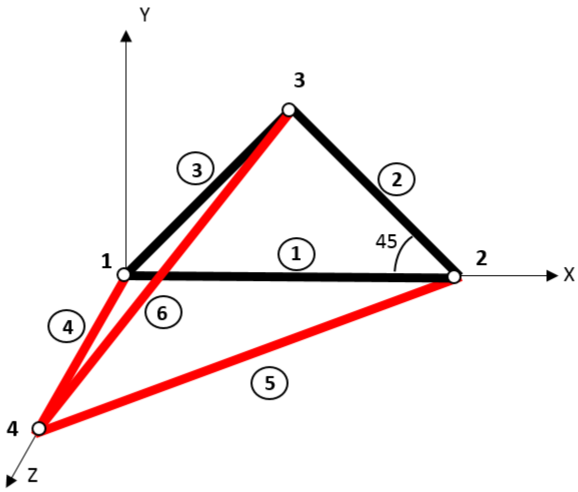
Transformation Matrix for a member of 3d truss: $\boldsymbol{T} = \begin{bmatrix} \cos\theta_X & 0 \\ \cos\theta_Y & 0 \\ \cos\theta_Z & 0 \\ 0 & \cos\theta_X \\ 0 & \cos\theta_Y \\ 0 & \cos\theta_Z \end{bmatrix}$

Solution 2. Calculate stresses on each member of the following 3D truss structure (Node 3 is in XY plane).

$L_2 = L_3 = L_4 = 100\ mm$, $E = 200\ GPa$, Area: $A = 100\ mm^2$,

B.C's: $U_{1X} = U_{1Y} = U_{1Z} = U_{2Y} = U_{2Z} = U_{4X} = U_{4Y} = U_{4Z} = 0$

Load: 10 kN on node 3 along negative Y direction.



Element Connectivity		
Element No.	Nodes	
1	1	2
2	2	3
3	1	3
4	1	4
5	2	4
6	3	4

Coordinates of nodes

Node #	X	Y	Z
1	0	0	0
2	L_1	0	0
3	$\frac{L_1}{2}$	$\frac{L_1}{2}$	0
4	0	0	L_4

Stiffness of Element 1 (element coordinate frame)

$L_1 = \sqrt{L_2^2 + L_3^2} = \sqrt{100^2 + 100^2} = 141.42\ mm$

$$k_1 = \frac{E * A}{L_1} = \frac{(200000\ MPa) * (100\ mm^2)}{141.42\ mm} = 1.4142\ x\ 10^5\ \frac{N}{mm}$$

Stiffness of Element 2, 3 and 4 (element coordinate frame)

$$k_2 = \frac{E * A}{L_2} = \frac{(200000\ MPa) * (100\ mm^2)}{100\ mm} = 2.0\ x\ 10^5\ \frac{N}{mm}$$

$L_2 = L_3 = L_4$ ➔ $k_2 = k_3 = k_4$

Stiffness of Element 5 (element coordinate frame)

$L_5 = \sqrt{L_1^2 + L_4^2} = \sqrt{141.42^2 + 100^2} = 173.21\ mm$

$$k_5 = \frac{E * A}{L_5} = \frac{(200000\ MPa) * (100\ mm^2)}{173.21\ mm} = 1.1547\ x\ 10^5 \frac{N}{mm}$$

Stiffness of Element 6 (element coordinate frame)

$$L_6 = \sqrt{L_3^2 + L_4^2} = \sqrt{100^2 + 100^2} = 141.42\ mm$$

$$k_6 = \frac{E * A}{L_6} = \frac{(200000\ MPa) * (100\ mm^2)}{173.21\ mm} = 1.4142\ x\ 10^5 \frac{N}{mm}$$

Then, the coordinates of nodes are;

Node #	**X**	**Y**	**Z**
1	0	0	0
2	$141.42\ mm$	0	0
3	$70.71\ mm$	$70.71\ mm$	0
4	0	0	$100\ mm$

Element 1 stiffness matrix (Global Coordinate Frame):

From equations 2.19, 2.20 and 2.21 direction cosines can be found for each elements

$\cos\theta_X = \frac{X_j - X_i}{L}$ $\cos\theta_Y = \frac{Y_j - Y_i}{L}$ $\cos\theta_Z = \frac{Z_j - Z_i}{L}$

Local coordinate frame is from Node 1 to Node 2 (connectivity matrix). Therefore, $i = 1, j = 2$

$\cos\theta_{1X} = \frac{X_2 - X_1}{L_1} = 1$ $\cos\theta_{1Y} = \frac{Y_2 - Y_1}{L_1} = 0$ $\cos\theta_{1Z} = \frac{Z_2 - Z_1}{L_1} = 0$

From equation 2.23 element stiffness matrix can be found as;

$$[\boldsymbol{K}]^1 = 1.4142\ x\ 10^5 \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{N}{mm}$$

Element 2 stiffness matrix (Global Coordinate Frame):

Local coordinate frame is from Node 2 to Node 3

$\cos\theta_{2X} = \frac{X_3 - X_2}{L_2} = -0.7071$ $\cos\theta_{2Y} = \frac{Y_3 - Y_2}{L_2} = 0.7071$ $\cos\theta_{2Z} = \frac{Z_3 - Z_2}{L_2} = 0$

$$[\boldsymbol{K}]^2 = 2.0\ x\ 10^5 \begin{bmatrix} 0.5 & -0.5 & 0 & -0.5 & 0.5 & 0 \\ -0.5 & 0.5 & 0 & 0.5 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 0.5 & 0 & 0.5 & -0.5 & 0 \\ 0.5 & -0.5 & 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{N}{mm}$$

Element 3 stiffness matrix (Global Coordinate Frame):

Local coordinate frame is from Node 1 to Node 3

$\cos\theta_{3X} = \frac{X_3 - X_1}{L_3} = 0.7071$ $\cos\theta_{3Y} = \frac{Y_3 - Y_1}{L_3} = 0.7071$ $\cos\theta_{3Z} = \frac{Z_3 - Z_1}{L_3} = 0$

$$[\boldsymbol{K}]^3 = 2.0\ x\ 10^5 \begin{bmatrix} 0.5 & 0.5 & 0 & -0.5 & -0.5 & 0 \\ 0.5 & 0.5 & 0 & -0.5 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{N}{mm}$$

Element 4 stiffness matrix (Global Coordinate Frame):

Local coordinate frame is from Node 1 to Node 4

$\cos\theta_{4X} = \frac{X_4 - X_1}{L_4} = 0$ $\cos\theta_{4Y} = \frac{Y_4 - Y_1}{L_4} = 0$ $\cos\theta_{4Z} = \frac{Z_4 - Z_1}{L_4} = 1$

$$[\boldsymbol{K}]^4 = 2.0\ x\ 10^5 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \frac{N}{mm}$$

Element 5 stiffness matrix (Global Coordinate Frame):

Local coordinate frame is from Node 2 to Node 4

$\cos\theta_{5X} = \frac{X_4 - X_2}{L_5} = -0.817$ $\cos\theta_{5Y} = \frac{Y_4 - Y_2}{L_5} = 0$ $\cos\theta_{5Z} = \frac{Z_4 - Z_2}{L_5} = 0.577$

$$[\boldsymbol{K}]^5 = 1.1547\ x\ 10^5 \begin{bmatrix} 0.6675 & 0 & -0.4714 & -0.6675 & 0 & 0.4717 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.4714 & 0 & 0.3329 & 0.4714 & 0 & -0.3329 \\ -0.6675 & 0 & 0.4714 & 0.6675 & 0 & -0.4717 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4714 & 0 & -0.3329 & -0.4714 & 0 & 0.3329 \end{bmatrix} \frac{N}{mm}$$

Element 6 stiffness matrix (Global Coordinate Frame):

Local coordinate frame is from Node 3 to Node 4

$\cos\theta_{6X} = \frac{X_4 - X_3}{L_6} = -0.5$ $\cos\theta_{6Y} = \frac{Y_4 - Y_3}{L_6} = -0.5$ $\cos\theta_{6Z} = \frac{Z_4 - Z_3}{L_6} = \frac{L_4}{L_6} = 0.7071$

$$[\boldsymbol{K}]^6 = 1.4142\ x\ 10^5 \begin{bmatrix} 0.25 & 0.25 & -0.3536 & -0.25 & -0.25 & 0.3536 \\ 0.25 & 0.25 & -0.3536 & -0.25 & -0.25 & 0.3536 \\ -0.3536 & -0.3536 & 0.5 & 0.3536 & 0.3536 & -0.5 \\ -0.25 & -0.25 & 0.3536 & 0.25 & 0.25 & -0.3536 \\ -0.25 & -0.25 & 0.3536 & 0.25 & 0.25 & -0.3536 \\ 0.3536 & 0.3536 & -0.5 & -0.3536 & -0.3536 & 0.5 \end{bmatrix} \frac{N}{mm}$$

ASSEMBLY OF GLOBAL STIFFNESS MATRIX, [K] =

$$
\begin{array}{ccccccccccccc}
U_{1X} & U_{1Y} & U_{1Z} & U_{2X} & U_{2Y} & U_{2Z} & U_{3X} & U_{3Y} & U_{3Z} & U_{4X} & U_{4Y} & U_{4Z} & \\
K^1_{11}+K^3_{11}+K^4_{11} & K^1_{12}+K^3_{12}+K^4_{12} & K^1_{13}+K^3_{13}+K^4_{13} & K^1_{14} & K^1_{15} & K^1_{16} & K^3_{14} & K^3_{15} & K^3_{16} & K^4_{14} & K^4_{15} & K^4_{16} & U_{1X} \\
K^1_{21}+K^3_{21}+K^4_{21} & K^1_{22}+K^3_{22}+K^4_{22} & K^1_{23}+K^3_{23}+K^4_{23} & K^1_{24} & K^1_{25} & K^1_{26} & K^3_{24} & K^3_{25} & K^3_{26} & K^4_{24} & K^4_{25} & K^4_{26} & U_{1Y} \\
K^1_{31}+K^3_{31}+K^4_{31} & K^1_{32}+K^3_{32}+K^4_{32} & K^1_{33}+K^3_{33}+K^4_{33} & K^1_{34} & K^1_{35} & K^1_{36} & K^3_{34} & K^3_{35} & K^3_{36} & K^4_{34} & K^4_{35} & K^4_{36} & U_{1Z} \\
K^1_{41} & K^1_{42} & K^1_{43} & K^1_{44}+K^2_{11}+K^5_{11} & K^1_{45}+K^2_{12}+K^5_{12} & K^1_{46}+K^2_{13}+K^5_{13} & K^2_{14} & K^2_{15} & K^2_{16} & K^5_{14} & K^5_{15} & K^5_{16} & U_{2X} \\
K^1_{51} & K^1_{52} & K^1_{53} & K^1_{54}+K^2_{21}+K^5_{21} & K^1_{55}+K^2_{22}+K^5_{22} & K^1_{56}+K^2_{23}+K^5_{23} & K^2_{24} & K^2_{25} & K^2_{26} & K^5_{24} & K^5_{25} & K^5_{26} & U_{2Y} \\
K^1_{61} & K^1_{62} & K^1_{63} & K^1_{64}+K^2_{31}+K^5_{31} & K^1_{65}+K^2_{32}+K^5_{32} & K^1_{66}+K^2_{33}+K^5_{33} & K^2_{34} & K^2_{35} & K^2_{36} & K^5_{34} & K^5_{35} & K^5_{36} & U_{2Z} \\
K^3_{41} & K^3_{42} & K^3_{43} & K^2_{41} & K^2_{42} & K^2_{43} & K^2_{44}+K^3_{44}+K^6_{11} & K^2_{45}+K^3_{45}+K^6_{12} & K^2_{46}+K^3_{46}+K^6_{13} & K^6_{14} & K^6_{15} & K^6_{16} & U_{3X} \\
K^3_{51} & K^3_{52} & K^3_{53} & K^2_{51} & K^2_{52} & K^2_{53} & K^2_{54}+K^3_{54}+K^6_{21} & K^2_{55}+K^3_{55}+K^6_{22} & K^2_{56}+K^3_{56}+K^6_{23} & K^6_{24} & K^6_{25} & K^6_{26} & U_{3Y} \\
K^3_{61} & K^3_{62} & K^3_{63} & K^2_{61} & K^2_{62} & K^2_{63} & K^2_{64}+K^3_{64}+K^6_{31} & K^2_{65}+K^3_{65}+K^6_{32} & K^2_{66}+K^3_{66}+K^6_{33} & K^6_{34} & K^6_{35} & K^6_{36} & U_{3Z} \\
K^4_{41} & K^4_{42} & K^4_{43} & K^5_{41} & K^5_{42} & K^5_{43} & K^6_{41} & K^6_{42} & K^6_{43} & K^4_{44}+K^5_{44}+K^6_{44} & K^4_{45}+K^5_{45}+K^6_{45} & K^4_{46}+K^5_{46}+K^6_{46} & U_{4X} \\
K^4_{51} & K^4_{52} & K^4_{53} & K^5_{51} & K^5_{52} & K^5_{53} & K^6_{51} & K^6_{52} & K^6_{53} & K^4_{54}+K^5_{54}+K^6_{54} & K^4_{55}+K^5_{55}+K^6_{55} & K^4_{56}+K^5_{56}+K^6_{56} & U_{4Y} \\
K^4_{61} & K^4_{62} & K^4_{63} & K^5_{61} & K^5_{62} & K^5_{63} & K^6_{61} & K^6_{62} & K^6_{63} & K^4_{64}+K^5_{64}+K^6_{64} & K^4_{65}+K^5_{65}+K^6_{65} & K^4_{66}+K^5_{66}+K^6_{66} & U_{4Z}
\end{array}
$$

Apply B.C.'s and obtain reduced global stiffness matrix

$$U_{1X} = U_{1Y} = U_{1Z} = U_{2Y} = U_{2Z} = U_{4X} = U_{4Y} = U_{4Z} = 0$$

$$
[\boldsymbol{K_{red}}] = \begin{array}{ccccc}
U_{2X} & U_{3X} & U_{3Y} & U_{3Z} & \\
K^1_{44}+K^2_{11}+K^5_{11} & K^2_{14} & K^2_{15} & K^2_{16} & U_{2X} \\
K^2_{41} & K^2_{44}+K^3_{44}+K^6_{11} & K^2_{45}+K^3_{45}+K^6_{12} & K^2_{46}+K^3_{46}+K^6_{13} & U_{3X} \\
K^2_{51} & K^2_{54}+K^3_{54}+K^6_{21} & K^2_{55}+K^3_{55}+K^6_{22} & K^2_{56}+K^3_{56}+K^6_{23} & U_{3Y} \\
K^2_{61} & K^2_{64}+K^3_{64}+K^6_{31} & K^2_{65}+K^3_{65}+K^6_{32} & K^2_{66}+K^3_{66}+K^6_{33} & U_{3Z}
\end{array}
$$

$$
[\boldsymbol{K_{red}}] = 1x10^4 \begin{bmatrix} 31.85 & -10 & 10 & 0 \\ -10 & 23.54 & 3.54 & -5 \\ 10 & 3.54 & 23.54 & -5 \\ 0 & -5 & -5 & 7.07 \end{bmatrix} \frac{N}{mm}
$$

Solve for unknown displacements

$$\begin{Bmatrix} U_{2X} \\ U_{3X} \\ U_{3Y} \\ U_{3Z} \end{Bmatrix} = [\boldsymbol{K_{red}}]^{-1} \begin{Bmatrix} F_{2X} \\ F_{3X} \\ F_{3Y} \\ F_{3Z} \end{Bmatrix}$$

$$\begin{Bmatrix} U_{2X} \\ U_{3X} \\ U_{3Y} \\ U_{3Z} \end{Bmatrix} = [\boldsymbol{K_{red}}]^{-1} \begin{Bmatrix} 0 \\ 0 \\ -10000 \\ 0 \end{Bmatrix} \cong \begin{Bmatrix} 0.023 \\ 0.011 \\ -0.061 \\ -0.035 \end{Bmatrix} mm$$

Stress on Element 1

Nodal displacements on element coordinate frame

$$\boldsymbol{u} = \boldsymbol{T^T} * \boldsymbol{U} \Rightarrow \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta_{1X} & \cos\theta_{1Y} & \cos\theta_{1Z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_{1X} & \cos\theta_{1Y} & \cos\theta_{1Z} \end{bmatrix} * \begin{Bmatrix} U_{1X} \\ U_{1Y} \\ U_{1Z} \\ U_{2X} \\ U_{2Y} \\ U_{2Z} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.023 \end{Bmatrix} mm$$

$$\sigma_1 = E * \varepsilon_1 = E * \frac{u_2 - u_1}{L_1} = 200000 * \frac{0.023}{141.42} = 32.5\ MPa$$

Stress on Element 2

Nodal displacements on element coordinate frame

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta_{2X} & \cos\theta_{2Y} & \cos\theta_{2Z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_{2X} & \cos\theta_{2Y} & \cos\theta_{2Z} \end{bmatrix} * \begin{Bmatrix} U_{2X} \\ U_{2Y} \\ U_{2Z} \\ U_{3X} \\ U_{3Y} \\ U_{3Z} \end{Bmatrix} = \begin{Bmatrix} -0.016 \\ -0.052 \end{Bmatrix} mm$$

$$\sigma_2 = E * \varepsilon_2 = E * \frac{u_3 - u_2}{L_2} = 200000 * \frac{(-0.052 + 0.016)}{100} = -72\ MPa$$

Stress on Element 3

Nodal displacements on element coordinate frame

$$\begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta_{3X} & \cos\theta_{3Y} & \cos\theta_{3Z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_{3X} & \cos\theta_{3Y} & \cos\theta_{3Z} \end{bmatrix} * \begin{Bmatrix} U_{1X} \\ U_{1Y} \\ U_{1Z} \\ U_{3X} \\ U_{3Y} \\ U_{3Z} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.035 \end{Bmatrix} mm$$

$$\sigma_3 = E * \varepsilon_3 = E * \frac{u_3 - u_1}{L_3} = 200000 * \frac{-0.035}{100} = -70\ MPa$$

Stress on Element 4

Nodal displacements on element coordinate frame

$$\begin{Bmatrix} u_1 \\ u_4 \end{Bmatrix} = \begin{bmatrix} \cos\theta_{4X} & \cos\theta_{4Y} & \cos\theta_{4Z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_{4X} & \cos\theta_{4Y} & \cos\theta_{4Z} \end{bmatrix} * \begin{Bmatrix} U_{1X} \\ U_{1Y} \\ U_{1Z} \\ U_{4X} \\ U_{4Y} \\ U_{4Z} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} mm$$

$$\sigma_4 = E * \varepsilon_4 = E * \frac{u_4 - u_1}{L_4} = 200000 * \frac{0}{100} = 0\ MPa$$

Stress on Element 5

Nodal displacements on element coordinate frame

$$\begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{bmatrix} \cos\theta_{5X} & \cos\theta_{5Y} & \cos\theta_{5Z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_{5X} & \cos\theta_{5Y} & \cos\theta_{5Z} \end{bmatrix} * \begin{Bmatrix} U_{2X} \\ U_{2Y} \\ U_{2Z} \\ U_{4X} \\ U_{4Y} \\ U_{4Z} \end{Bmatrix} = \begin{Bmatrix} -0.019 \\ 0 \end{Bmatrix} mm$$

$$\sigma_5 = E * \varepsilon_5 = E * \frac{u_4 - u_2}{L_5} = 200000 * \frac{(0+0.019)}{173.21} = 21.9\ MPa$$

Stress on Element 6

Nodal displacements on element coordinate frame

$$\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} \cos\theta_{6X} & \cos\theta_{6Y} & \cos\theta_{6Z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_{6X} & \cos\theta_{6Y} & \cos\theta_{6Z} \end{bmatrix} * \begin{Bmatrix} U_{3X} \\ U_{3Y} \\ U_{3Z} \\ U_{4X} \\ U_{4Y} \\ U_{4Z} \end{Bmatrix} \approx \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} mm$$

$$\sigma_6 = E * \varepsilon_6 = E * \frac{u_4 - u_3}{L_6} = 200000 * \frac{0}{141.42} \approx 0\ MPa$$