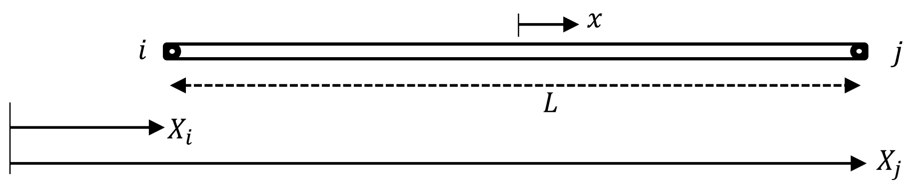
ME 402 – INTRODUCTION TO FINITE ELEMENT ANALYSIS (2014-2015 Spring)

HOMEWORK #2 - Solutions

Solution 1. Develop the shape functions for a linear element using the local coordinate x located at the **mid-point** of the element ($S_i(x) = ?,\ S_j(x) = ?$).



Displacement at an arbitrary point on the element can be estimated by a first order (linear) polynomial:

$U^{(e)}(x)$, where x is from $-L/2$ to $+L/2$

This polynomial should be satisfied at the end points, $i\ and\ j$

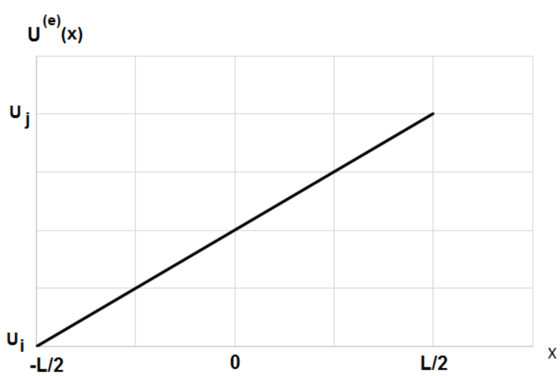
Equation at the i^{th} node: $x = -L/2$

$$U^{(e)}\left(-\frac{L}{2}\right) = U_i$$

Equation at the j^{th} node: $x = L/2$

$$U^{(e)}\left(\frac{L}{2}\right) = U_j$$

Linear polynomial can be plotted as follows;



Then the equation can be written as follows

$$U^{(e)}(x) = \left(\frac{U_j - U_i}{L}\right)x + \left(\frac{U_j + U_i}{2}\right)$$

$$U^{(e)}(x) = U_i\left(\frac{1}{2} - \frac{x}{L}\right) + U_j\left(\frac{1}{2} + \frac{x}{L}\right) = U_i * S_i(x) + U_j * S_j(x)$$

$$S_i(x) = \frac{1}{2} - \frac{x}{L} \quad and \quad S_j(x) = \frac{1}{2} + \frac{x}{L}$$

Solution 2. Evaluate the integral $\int_1^5 (x^3 + 6x^2 + 3x - 2)dx$

$f(x) = x^3 + 6x^2 + 3x - 2$

Firstly, the integral boundaries should be converted into -1 to $+1$ by converting variable x to r as follows;

$$x = \frac{x_2 + x_1}{2} + \frac{x_2 - x_1}{2} r = \frac{5+1}{2} + \frac{5-1}{2} r = 3 + 2r$$

Then, $dx = 2dr$

$$I = \int_1^5 (x^3 + 6x^2 + 3x - 2)dx = \int_{-1}^{1} ((3+2r)^3 + 6(3+2r)^2 + 3(3+2r) - 2) * 2 * dr$$

$$I = \int_{-1}^{1} 2((3+2r)^3 + 61 + 78r + 24r^2)\, dr$$

Let's define another function, $g(r) = 2((3+2r)^3 + 61 + 78r + 24r^2)$

Then, the integral take the following form

$$I = \int_{-1}^{1} g(r)\, dr$$

(a) Gauss-Legendre two-point sampling

$$I = \sum_{i=1}^{2} g(r_i) * w_i$$

Where, the sampling points and weighting factors for two point integration are as follows

$$r_1 = -0.577350269, \quad r_2 = -0.577350269$$

$$w_1 = 1 \quad , \quad w_2 = 1$$

$$I = g(r_1) * w_1 + g(r_2) * w_2$$

$$I = g(-0.577350269) * 1 + g(0.577350269) * 1$$

$$I = 60.5 * 1 + 371.5 * 1 = 432$$

(b) Gauss-Legendre three-point sampling

$$I = \sum_{i=1}^{3} g(r_i) * w_i$$

Where, the sampling points and weighting factors for two point integration are as follows

$$r_1 = -0.774596669, \quad r_2 = 0, \quad r_3 = 0.774596669$$

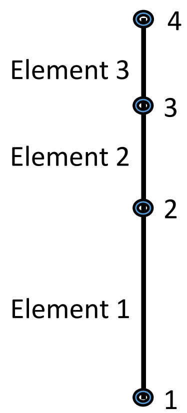
$$w_1 = 0.55555556 \quad , \quad w_2 = 0.88888889, \quad w_3 = 0.55555556$$

$$I = g(r_1) * w_1 + g(r_2) * w_2 + g(r_3) * w_3$$

$$I = g(-0.774596669) * 0.55555556 + g(0) * 0.88888889 + g(0.774596669) * 0.55555556$$

$$I = 20.039 + 156.44 + 255.52 = 432$$

Solution 3. Solve Problem 16 (Chapter 4 - "Finite Element Analysis, Theory and Application with ANSYS", Prentice-Hall, 1999 [TA347.F5M62 1999]). Report your results in metric units.



$A_1 = 2.95\ in^2 = 1903\ mm^2, A_2 = 2.15\ in^2 = 1387\ mm^2, A_3 = 0.75\ in^2 = 484\ mm^2$

$L_1 = 10\ ft = 3048\ mm,\ L_2 = 5\ ft = 1524\ mm, L_3 = 5\ ft = 1524\ mm$

$E = 29\ x10^6\ \frac{lb}{in^2} = 200\ GPa$

Boundary Condition: $U_1 = 0$

Loads:

$F_2 = -200\ lb * g = -889.9\ N$

$F_3 = -150\ lb * g = -667.5\ N$

$F_4 = -100\ lb * g = -445.0\ N$

Selected element connectivity matrix

Element #	**1st Node**	**2nd Node**
1	**1**	**2**
2	**2**	**3**
3	**3**	**4**

Element 1 Stiffness Matrix

$$k_1 = \frac{E * A_1}{L_1} = \frac{200000 * 1903}{3048} = 1.2487\ x\ 10^5\ \frac{N}{mm}$$

$$\boldsymbol{K}^{(1)} = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1.2487\ x\ 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{N}{mm}$$

Element 2 Stiffness Matrix

$$k_2 = \frac{E * A_2}{L_2} = \frac{200000 * 1387}{1524} = 1.8202\ x\ 10^5\ \frac{N}{mm}$$

$$\boldsymbol{K}^{(2)} = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1.8202\ x\ 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{N}{mm}$$

Element 3 Stiffness Matrix

$$k_3 = \frac{E * A_3}{L_3} = \frac{200000 * 484}{1524} = 6.3517\ x\ 10^4\ \frac{N}{mm}$$

$$\boldsymbol{K}^{(3)} = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 6.3517\ x\ 10^4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{N}{mm}$$

Assembly of Global Stiffness Matrix

$$[\boldsymbol{K}] = \begin{matrix} U_1 & U_2 & U_3 & U_4 & \\ K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & U_1 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & 0 & U_2 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} & U_3 \\ 0 & 0 & K_{21}^{(3)} & K_{22}^{(3)} & U_4 \end{matrix}$$

$$[\boldsymbol{K}] = \begin{matrix} U_1 & U_2 & U_3 & U_4 & \\ k_1 & -k_1 & 0 & 0 & U_1 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & U_2 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & U_3 \\ 0 & 0 & -k_3 & k_3 & U_4 \end{matrix}$$

$$[\boldsymbol{K}] = \begin{bmatrix} 1.2487\ x\ 10^5 & -1.2487\ x\ 10^5 & 0 & 0 \\ -1.2487\ x\ 10^5 & 3.0689\ x\ 10^5 & -1.8202\ x\ 10^5 & 0 \\ 0 & -1.8202\ x\ 10^5 & 2.4554\ x\ 10^5 & -6.3517\ x\ 10^4 \\ 0 & 0 & -6.3517\ x\ 10^4 & 6.3517\ x\ 10^4 \end{bmatrix} \frac{N}{mm}$$

Apply Boundary Condition: $U_1 = 0$

$$[\boldsymbol{K_{red}}] = \begin{bmatrix} 306890 & -182020 & 0 \\ -182020 & 245540 & -63517 \\ 0 & -63517 & 63517 \end{bmatrix} \frac{N}{mm}$$

Solve for unknown displacements

$$\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = [\boldsymbol{K_{red}}]^{-1} \begin{Bmatrix} F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = [\boldsymbol{K_{red}}]^{-1} \begin{Bmatrix} -889.9 \\ -667.5 \\ -445.0 \end{Bmatrix} \cong \begin{Bmatrix} -0.016 \\ -0.022 \\ -0.029 \end{Bmatrix} mm$$

Stress on Element 1

$$\sigma_1 = E * \varepsilon_1 = E * \frac{U_2 - U_1}{L_1} = 200000 * \frac{-0.016}{3048} = -1.05\ MPa$$

Stress on Element 2

$$\sigma_2 = E * \varepsilon_2 = E * \frac{U_3 - U_2}{L_2} = 200000 * \frac{-0.022 - (-0.016)}{1524} = -0.79\ MPa$$

Stress on Element 3

$$\sigma_3 = E * \varepsilon_3 = E * \frac{U_4 - U_3}{L_3} = 200000 * \frac{-0.029 - (-0.022)}{1524} = -0.92\ MPa$$